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LARGE CONTRACTION AND EXISTENCE OF PERIODIC SOLUTIONS IN INFINITE DELAY VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

In this research, we make use of the concept of large contraction and Krasnoselskii fixed point theorem to show that the totally nonlinear infinite delay Volterra integro-differential equation

$$x'(t) = -a(t)h(x(t)) + \int_{-\infty}^{t} B(t, s)g(x(s))ds + p(t),$$

has a periodic solution. The need for the use of large contraction arises from the nonlinear term $a(t)h(x(t))$. Several examples will be provided as illustration of our results.

1. Introduction

In this paper, we use a modified version of Krasnoselskii's fixed point theorem and show the highly nonlinear infinite delay Volterra integro-differential equation

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$$x'(t) = -a(t)h(x(t)) + \int_{-\infty}^{t} B(t, s)g(x(s))ds + p(t), \tag{1.1}$$

has a periodic solution. Throughout this paper, we assume all functions are continuous on their respective domains.

Since we are dealing with the existence of periodic solutions of Equation (1.1), it is natural to ask that for the least positive real number T, we have

$$a(t+T) = a(t),\ p(t+T) = p(t), \text{ and } B(t+T, s+T) = B(t, s), \tag{1.2}$$

for all $t \in \mathbb{R}$. Since Equation (1.1) is totally nonlinear, to invert it into an integral equation problem, we will have to add and subtract a linear term. For some particular functions $h(x)$, this process destroys the traditional contraction property for one of the mappings in Krasnoselskii's theorem. But the process replaces it with what is called a "large contraction". For more on the existence of periodic solutions, we refer the readers to [2], [3], [4], and the references therein. Next, we state Krasnoselskii's fixed point theorem. For its proof, we refer the readers to [6].

Theorem 1 (Krasnoselskiĭ)**.** *Let $\mathcal{M}$ be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathcal{M}$, *implies* $Ax + By \in \mathcal{M}$,

(ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

Concerning the terminology of compact mapping used in this theorem, we mean the following. Let A be a mapping from a set $\mathcal{M}$ into a topological space X. If $A(\mathcal{M})$ is contained in a compact subset of X, we say that A is compact.

Definition 1. Let $(\mathcal{M}, d)$ be a metric space and $B : \mathcal{M} \to \mathcal{M}$. B is said to be *large contraction* if $\phi, \varphi \in \mathcal{M}$, with $\phi \neq \varphi$, then $d(B\phi, B\varphi) \leq d(\phi, \varphi)$ and if for all $\varepsilon > 0$, there exists a $\delta < 1$ such that

$$[\phi, \varphi \in M, d(\phi, \varphi) \geq \varepsilon] \Rightarrow d(B\phi, B\varphi) \leq \delta d(\phi, \varphi).$$

The next theorem is appropriate for our equation since it requires a large contraction instead of contraction.

Theorem 2. *Let $\mathcal{M}$ be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathcal{M}$, *implies* $Ax + By \in \mathcal{M}$,

(ii) *A is compact and continuous*,

(iii) *B is a large contraction mapping.*

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

We shall see later that the concept of large contraction in Theorem 2 is necessary when $h(x(t)) = x^5(t)$.

2. Existence of Periodic Solutions

Define $P_T = \{\varphi \in C(\mathbb{R}, \mathbb{R}) : \varphi(t + T) = \varphi(t)\}$, where $C(\mathbb{R}, \mathbb{R})$ is the space of all real valued continuous functions on $\mathbb{R}$. Then P_T is a Banach space, when endowed with the supremum norm

$$\|x\| = \sup_{t\in[0, T]} |x(t)| = \sup_{t\in\mathbb{R}} |x(t)|.$$

The next lemma is essential to the construction of our mapping that is required for the application of Theorem 2. To have a well behaved mapping, we must assume that

$$\int_0^T a(s)ds \neq 0, \tag{2.1}$$

throughout this section. Let the mapping H be defined by

$$H(x(u)) = x(u) - h(x(u)). \tag{2.2}$$

Lemma 1. *Assume* (2.1). *If* $x \in P_T$, *then* $x(t)$ *is a solution of Equation* (1.1), *if and only if*

$$x(t) = \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} \left(a(u)H(x(u)) + \int_{-\infty}^{u} B(u, s)g(x(s))ds + p(u) \right) du. \tag{2.3}$$

Proof. Let $x \in P_T$ be a solution of (1.1). We rewrite (1.1) in the form

$$x'(t) + a(t)x(t) = a(t)H(x(u)) + \int_{-\infty}^{t} B(t, s)g(x(s))ds + p(t).$$

Next, we multiply both sides of the resulting equation with $e^{\int_0^t a(s)ds}$, and then integrate from t to $t + T$ to obtain

$$x(t + T)e^{\int_0^{t+T} a(s)ds} - x(t)e^{\int_0^t a(s)ds}$$

$$= \int_t^{t+T} \left[a(u)H(x(u)) + \int_{-\infty}^{u} B(u, s)g(x(s))ds + p(u) \right] e^{\int_0^u a(s)ds} du.$$

Using the fact that $x(t + T) = x(t)$ and $e^{\int_t^{t+T} a(s)ds} = e^{\int_0^T a(s)ds}$, the above expression can be put in the form of Equation (2.3). The proof is complete by reversing every step. □

First, we note that for $t \in [0, T]$ and $u \in [t, t + T]$, we have

$$\frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} \le \frac{e^{\int_0^{2T} |a(s)|ds}}{\left| 1 - e^{-\int_0^T a(s)ds} \right|} := M. \tag{2.4}$$

Let J be a positive constant. Define the set

$$\mathbb{M}_J = \{\varphi \in P_T : \|\varphi\| \leq J\}. \tag{2.5}$$

Obviously, $\mathbb{M}_J$ is bounded and convex subset of the Banach space P_T. Let the map $A : \mathbb{M}_J \to P_T$ be defined by

$$(A\varphi)(t) = \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} \left[\int_{-\infty}^u B(u, s)g(\varphi(s))ds + p(u)\right]du, \tag{2.6}$$

for $t \in \mathbb{R}$. In a similar way, we set the map $B : \mathbb{M}_J \to P_T$

$$(B\psi)(t) = \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} a(u)H(\psi(u))du,\ t \in \mathbb{R}. \tag{2.7}$$

It is clear from (2.6) and (2.7) that $A\phi$ and $B\psi$ are T-periodic in t.

We assume that $g(x)$ satisfies local Lipschitz condition in x, i.e., there is a positive constant k such that

$$|g(z) - g(w)| \leq k\|z - w\|, \text{ for } z, w \in \mathbb{M}_J. \tag{2.8}$$

Then for $\varphi \in \mathbb{M}_J$, we obtain the following:

$$\begin{aligned} |g(\varphi(t))| &= |g(\varphi(t)) - g(0) + g(0)| \\ &\leq |g(\varphi(t)) - g(t, 0)| + |g(t, 0)| \\ &\leq kJ + |g(0)|. \end{aligned} \tag{2.9}$$

For simplicity, we let

$$\eta := \left|(1 - e^{-\int_0^T a(s)ds})^{-1}\right|.$$

Lemma 2. *Suppose that there exists a positive constant L such that*

$$\int_{-\infty}^t |B(t, s)|ds \leq L, \tag{2.10}$$

then the mapping A, defined by (2.6), *is continuous in* $\varphi \in \mathbb{M}_J$.

Proof. Let $\phi, \varphi \in \mathbb{M}_J$. Then, from (2.6), we have

$$|A\phi(t) - A\varphi(t)| \le \Big|\int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} \int_{-\infty}^{u} B(u, s) g(\phi(s)) ds du$$

$$- \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} \int_{-\infty}^{u} B(u, s) g(\varphi(s)) ds du\Big|$$

$$\le \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} \int_{-\infty}^{u} |B(u, s)| |g(\phi(s)) - g(\varphi(s))| ds du$$

$$\le MLKT\|\phi - \varphi\|,$$

where M is given by (2.4). □

In next two results, we assume that for all $t \in \mathbb{R}$ and $\psi \in \mathbb{M}_J$

$$\eta \int_t^{t+T} [|a(u)| |H(\psi(u))| + L(KJ + |g(0)|)] e^{-\int_u^{t+T} a(s)ds} du \le J, \tag{2.11}$$

where J is defined by (2.5).

Lemma 3. *Assume* (1.2), (2.1), (2.8), (2.10), *and* (2.11). *Then A is continuous in $\varphi \in \mathbb{M}_J$ and maps $\mathbb{M}_J$ into a compact subset of $\mathbb{M}_J$.*

Proof. Let $\varphi \in \mathbb{M}_J$. Continuity of A in $\varphi \in \mathbb{M}_J$ follows from Lemma 2. Now, by (2.9), (2.10), and (2.11), we have $|(A\varphi)(t)| < J$. Thus, $A\varphi \in \mathbb{M}_J$. Let $\varphi_i \in \mathbb{M}$, $i = 1, 2, \ldots$. Then from the above discussion, we conclude that

$$\|A\varphi_j\| \le J.$$

This shows $A(\mathbb{M}_J)$ is uniformly bounded. Left to show that $A(\mathbb{M}_J)$ is equicontinuous. A differentiation of (2.6) with respect to t yields

$$
\begin{aligned}
|(A\varphi_j)'(t)| = \Big| &- a(t)(A\varphi_j)(t) \\
&+ \frac{1}{1 - e^{-\int_0^T a(s)ds}} \int_{-\infty}^{t+T} B(t, s) g(\phi(s)) ds \\
&+ \frac{e^{-\int_t^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} \int_{-\infty}^{t} B(t, s) g(\phi(s)) ds \Big| \\
&\leq \|a\| \|A\varphi_i\| + L(\eta + M)(kJ + |g(0)|) \leq Q,
\end{aligned}
$$

for some positive constant Q. Thus, the estimation on $|(A\varphi_i)'(t)|$ implies that $A(\mathbb{M}_J)$ is equicontinuous. Then using Arzela-Ascoli theorem, we obtain that A is a compact map. The proof is complete. □

Next result gives a relationship between the mappings H and B in the sense of large contraction.

Lemma 4. *Let a be a positive valued function. If H is a large contraction on $\mathbb{M}_J$, then so is the mapping B.*

Proof. If H is a large contraction on $\mathbb{M}_J$, then for $x, y \in \mathbb{M}_J$, with $x \neq y$, we have $\|Hx - Hy\| \leq \|x - y\|$. Hence,

$$
\begin{aligned}
|Bx(t) - By(t)| &\leq \int_t^{t+T} \frac{e^{-\int_u^{t+T} a(s)ds}}{1 - e^{-\int_0^T a(s)ds}} a(u) |H(x)(u) - H(y)(u)| du \\
&\leq \frac{\|x - y\|}{1 - e^{-\int_0^T a(s)ds}} \int_t^{t+T} e^{-\int_u^{t+T} a(s)ds} a(u) du \\
&= \|x - y\|.
\end{aligned}
$$

Taking supremum over the set $[0, T]$, we get that $\|Bx - By\| \leq \|x - y\|$. One may also show in a similar way that

$$\|Bx - By\| \le \delta\|x - y\|$$

holds, if we know the existence of a $0 < \delta < 1$ such that for all $\varepsilon > 0$

$$[x,\ y \in \mathbb{M}_J,\ \|x - y\| \ge \varepsilon] \Rightarrow \|Hx - Hy\| \le \delta\|x - y\|.$$

The proof is complete. □

Theorem 3. *Assume the hypotheses of Lemmas* 1-4. *If B is a large contraction on* $\mathbb{M}_J$, *then* (1.1) *has a periodic solution in* $\mathbb{M}_J$.

Proof. Let A and B be defined by (2.6) and (2.7), respectively. Then using (2.11) and the periodicity of A and B, we have for $\varphi, \psi \in \mathbb{M}_J$ that

$$A\varphi + B\psi : \mathbb{M}_J \to \mathbb{M}_J.$$

Hence an application of Krasnoselskii fixed point theorem implies the existence of a periodic solution in $\mathbb{M}$. This completes the proof. □

3. Examples

In this section, we provide several examples as application to our theory. In the first example, we explicitly define h and then show that the function H defines a large contraction. The next example can be found in [1]. For completeness, we give the full details here. First, we begin with the case $h(u) = u^5$.

Example 1. Let $\|.\|$ denote the supremum norm. If

$$\mathbb{M} = \left\{\phi : \phi \in C(\mathbb{R}, \mathbb{R}) \text{ and } \|\phi\| \le 5^{-1/4}\right\},$$

and $h(u) = u^5$, then the mapping H defined by (2.2) is a large contraction on the set $\mathbb{M}$.

Proof. For any reals a and b, we have the following inequalities:

$$0 \le (a + b)^4 = a^4 + b^4 + ab(4a^2 + 6ab + 4b^2),$$

and

$$-ab(a^2 + ab + b^2) \le \frac{a^4 + b^4}{4} + \frac{a^2b^2}{2} \le \frac{a^4 + b^4}{2}.$$

If $x, y \in \mathbb{M}$ with $x \neq y$, then $[x(t)]^4 + [y(t)]^4 < 1$. Hence, we arrive at

$$\begin{aligned}|H(u) - H(v)| &\le |u - v|\left|1 - \left(\frac{u^5 - v^5}{u - v}\right)\right| \\ &= |u - v|\,[1 - u^4 - v^4 - uv(u^2 + uv + v^2)] \\ &\le |u - v|\left[1 - \frac{(u^4 + v^4)}{2}\right] \le |u - v|, \qquad (3.1)\end{aligned}$$

where we use the notations $u = x(t)$ and $v = y(t)$ for brevity. Now, we are ready to show that H is a large contraction on $\mathbb{M}$. For a given $\varepsilon \in (0, 1)$, suppose $x, y \in \mathbb{M}$ with $\|x - y\| \ge \varepsilon$. There are two cases:

(a) $\frac{\varepsilon}{2} \le |x(t) - y(t)|$, for some $t \in \mathbb{R}$,

or

(b) $|x(t) - y(t)| \le \frac{\varepsilon}{2}$, for some $t \in \mathbb{R}$.

If $\frac{\varepsilon}{2} \le |x(t) - y(t)|$ for some $t \in \mathbb{R}$, then

$$(\varepsilon / 2)^4 \le |x(t) - y(t)|^4 \le 8(x(t)^4 + y(t)^4)$$

or

$$x(t)^4 + y(t)^4 \ge \frac{\varepsilon^4}{2^7}.$$

For all such t, we get by (3.1) that

$$|H(x(t)) - H(y(t))| \le |x(t) - y(t)|\left(1 - \frac{\varepsilon^4}{2^7}\right).$$

On the other hand, if $|x(t) - y(t)| \leq \frac{\varepsilon}{2}$ for some $t \in \mathbb{R}$, then along with (3.1), we find

$$|H(x(t)) - H(y(t))| \leq |x(t) - y(t)| \leq \frac{1}{2}\|x - y\|.$$

Hence, in both cases, we have

$$|H(x(t)) - H(y(t))| \leq \min\left\{1 - \frac{\varepsilon^4}{2^7}, \frac{1}{2}\right\}\|x - y\|.$$

Thus, H is a large contraction on the set $\mathbb{M}$ with $\delta = \min\left\{1 - \frac{\varepsilon^4}{2^7}, \frac{1}{2}\right\}$. The proof is complete. $\square$

Next, we make use of Example 1 and Theorem 3 to show that the totally nonlinear infinite delay Volterra integro-differential equation

$$x'(t) = -a(t)x^5(t) + \int_{-\infty}^{t} B(t, s)x^5(s)ds + p(t), \tag{3.2}$$

has a T-periodic solution. First, we assume that

$$4(5^{-5/4}) + \eta\int_{t}^{t+T}\left(5^{-5/4}\int_{-\infty}^{u}|B(u, s)|ds + |p(u)|\right)e^{-\int_u^{t+T} a(s)ds}du \leq 5^{-1/4}. \tag{3.3}$$

Example 2. Let the set $\mathbb{M}$ be a subset of P_T and defined as in Example 1. Assume (1.2), (2.1), and (2.8). Suppose $a(t) > 0$ for all t. If (3.3) holds for all $t \in \mathbb{R}$, then Equation (3.2) has a T-periodic solution in $\mathbb{M}$.

Proof. Set

$$g(x) = x^5,$$

and

$$h(\psi(u)) = \psi^5(u),$$

and define the mapping B as in (2.7). First, for $x \in \mathbb{M}$, we have

$$|x(t)|^5 \leq 5^{-5/4}.$$

Accordingly, define the two mappings A and B by (2.6) and (2.7), respectively. It is easy to see that

$$|H(x(t))| = |x(t) - x(t)^5| \leq 4(5^{-5/4}), \text{ for all } x \in \mathbb{M}.$$

For $\phi, \varphi \in \mathbb{M}$ and for $a(t) \geq 0$, we have

$$|(A\phi)(t) + (B\varphi)(t)|$$

$$\leq \frac{4(5^{-5/4})}{1 - e^{-\int_0^T a(s)ds}} \int_t^{t+T} a(u) e^{-\int_u^{t+T} a(s)ds} du$$

$$+ \eta \int_t^{t+T} \left(5^{-5/4} \int_{-\infty}^u |B(u, s)| ds + |p(u)|\right) e^{-\int_u^{t+T} a(s)ds} du$$

$$= 4(5^{-5/4}) + \eta \int_t^{t+T} \left(5^{-5/4} \int_{-\infty}^u |B(u, s)| ds + |p(u)|\right) e^{-\int_u^{t+T} a(s)ds} du$$

$$\leq 5^{-1/4} := J \text{ by (3.3).}$$

Thus, (2.11) is satisfied. Since a is assumed to be positive valued, we get by Lemma 4 that B is a large contraction on the set $\mathbb{M}$. □

Example 3. Let the set $\mathbb{M}$ be a subset of P_T and defined as in Example 1. Assume (1.2), (2.1), and (2.8). Suppose $a(t) > 0$ for all t. If there exists a positive constant D such that

$$D\left(5^{-5/4} \int_{-\infty}^t |B(t, s)| ds + |p(t)|\right) \leq a(t), \text{ for all } t \in \mathbb{R},$$

and

$$4(5^{-5/4}) + \frac{1}{D} \leq 5^{-1/4},$$

then (3.3) has a T-periodic solution in $\mathbb{M}$.

Proof. We already know that H defines a large contraction. For $\phi, \varphi \in \mathbb{M}$ and for $a(t) > 0$, we have

$$|(A\phi)(t) + (B\varphi)(t)|$$

$$\leq \frac{4(5^{-5/4})}{1 - e^{-\int_0^T a(s)ds}} \int_t^{t+T} a(u)\, e^{-\int_u^{t+T} a(s)ds} du$$

$$+ \frac{1}{1 - e^{-\int_0^T a(s)ds}} \int_t^{t+T} (5^{-5/4} \int_{-\infty}^{u} |B(u, s)| ds + |p(u)|) e^{-\int_u^{t+T} a(s)ds} du$$

$$= 4(5^{-5/4}) + \frac{1}{D} \int_t^{t+T} \frac{a(u)}{1 - e^{-\int_0^T a(s)ds}} e^{-\int_u^{t+T} a(s)ds} du$$

$$= 4(5^{-5/4}) + \frac{1}{D} \leq 5^{-1/4} := J,$$

which implies that (3.2) holds. The rest of the proof follows from Example 2. □

We note that in the paper of [1], the authors gave a theorem in which, they classify all type of functions h so that the function $H(x) = x - h(x)$ defines a large contraction.

References

[1] M. Adivar, M. Islam and Y. N. Raffoul, Existence of periodic and positive periodic solutions in totally nonlinear delay differential equations, submitted.

[2] E. R. Kaufmann and Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, J. Math. Anal. Appl. 319(1) (2006), 315-325.

[3] E. R. Kaufmann and Y. N. Raffoul, Periodicity and stability in neutral nonlinear dynamic equations with functional delay on a time scale, Electron. J. Differential Equations 27 (2007), 12.

[4] Y. N. Raffoul, Periodic solutions for neutral nonlinear differential equations with functional delay, Electron. J. Differential Equations 102 (2003), 1-7.

[5] Y. N. Raffoul, Positive periodic solutions in neutral nonlinear differential equations, E. J. Qualitative Theory of Diff. Equ. 16 (2007), 1-10.

[6] D. R. Smart, Fixed Points Theorems, Cambridge Univ. Press, Cambridge, 1980.

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